Multi-dimensional Central Limit Theorem

Outline

Consider a sequence of independent random processes $X_1(t), X_2(t), \cdots$ identical to some $X(t)$.

Assume $\bar{X}(t) = 0$.

Define the sum process $Z(t)$ as

$$Z(t) = \frac{X_1(t) + X_2(t) + \cdots + X_N(t)}{\sqrt{N}}$$

As $N \rightarrow \infty$, $Z(t)$ becomes a Gaussian random process.

$\{Z(t_i), Z(t_2), \cdots, Z(t_k)\}$ are jointly Gaussian for any $k$ and for any sampling instants.
Joint Characteristic Function of a Random Vector

**Definition.** For a $k$-dimensional random vector $X = (X_1, X_2, \ldots, X_k)$, define its joint characteristic function as
\[
\Phi_X(\omega_1, \omega_2, \ldots, \omega_k) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{i\omega_1 x_1} e^{i\omega_2 x_2} \cdots e^{i\omega_k x_k} f_X(x_1, x_2, \ldots, x_k) \, dx_1 \, dx_2 \cdots dx_k
\]
where $f_X(x_1, x_2, \ldots, x_k)$ is the joint pdf of $X$.

Using the expectation notation,
\[
\Phi_X(\omega_1, \omega_2, \ldots, \omega_k) = e^{i(\omega_1 X_1 + \omega_2 X_2 + \cdots + \omega_k X_k)}
\]
(g1)

When the random variables $\{X_i\}$ are statistically independent,
\[
\Phi_X(\omega_1, \omega_2, \ldots, \omega_k) = e^{i\omega_1 X_1} e^{i\omega_2 X_2} \cdots e^{i\omega_k X_k}
\]

In the one-dimensional case,
\[
\Phi_X(\omega) = e^{i\omega X}
\]

Our vectors are row vectors. Using matrix notation,

Let $\omega = (\omega_1, \omega_2, \ldots, \omega_k)$ and $X = (X_1, X_2, \ldots, X_k)$.

Then
\[
\omega X^T = \omega_1 X_1 + \omega_2 X_2 + \cdots + \omega_k X_k
\]
and eq. (g1) is written as
\[
\Phi_X(\omega) = e^{i\omega X^T}
\]
(g2)
Joint Characteristic Function of a Subset

Let \( X = (X_1, X_2, \ldots, X_k) \).

Consider a subset of the random variables, say, \( (X_1, X_2, \ldots, X_\ell) \), \( \ell < k \).

The joint characteristic function of \( (X_1, X_2, \ldots, X_\ell) \) can be found easily from the joint characteristic function of \( (X_1, X_2, \ldots, X_k) \):

\[
\Phi_{X_1, X_2, \ldots, X_\ell}(\omega_1, \omega_2, \ldots, \omega_\ell) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{i\omega_1 x_1} e^{i\omega_2 x_2} \cdots e^{i\omega_\ell x_\ell} \Phi_{X_1, X_2, \ldots, X_k}(x_1, x_2, \ldots, x_\ell, x_{\ell+1}, \ldots, x_k) \, dx_1 dx_2 \cdots dx_\ell dx_{\ell+1} \cdots dx_k
\]

\[
= \Phi_{X_1, X_2, \ldots, X_{\ell+1}, \ldots, X_k}(\omega_1, \omega_2, \ldots, \omega_\ell, \omega_{\ell+1} = 0, \ldots, \omega_k = 0)
\]

Example

\[
\Phi_{X_1, X_2}(\omega_1, \omega_2) = \Phi_{X_1, X_2, X_3}(\omega_1, \omega_2 = 0, \omega_3)
\]

\[
\Phi_{X_1}(\omega_1) = \Phi_{X_1, X_2, X_3}(\omega_1, \omega_2 = 0, \omega_3 = 0)
\]
**Covariance Matrix of a Random Vector**

Consider a $k$-dimensional random vector $X = (X_1, X_2, \ldots, X_k)$.

Define

$$
\lambda_{ij} = \text{cov}(X_i, X_j) = \frac{X_i X_j}{n} - \overline{X_i} \overline{X_j}
$$

and

$$
\Lambda_X = \begin{pmatrix}
\lambda_{11} & \lambda_{12} & \cdots & \lambda_{1k} \\
\lambda_{21} & \lambda_{22} & \cdots & \lambda_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{k1} & \lambda_{k2} & \cdots & \lambda_{kk}
\end{pmatrix}
$$

$\Lambda_X$ is referred to as the covariance matrix of the random vector $X$.

**property**

The diagonal elements of the covariance matrix is

$$
\lambda_{jj} = \text{cov}(X_j, X_j) = \sigma_j^2
$$

that is, the variance of $X_j$.

**property**

For $i \neq j$,

$$
\lambda_{ij} = \text{cov}(X_i, X_j) = \frac{X_i X_j}{n} - \overline{X_i} \overline{X_j} = \lambda_{ji}
$$

that is, $\Lambda_X$ is symmetrical.

**property**

The correlation coefficient is

$$
\rho_{ij} = \frac{\text{cov}(X_i, X_j)}{\sigma_i \sigma_j} = \frac{\lambda_{ij}}{\sigma_i \sigma_j}
$$

Thus

$$
\lambda_{ij} = \rho_{ij} \sigma_i \sigma_j
$$
property

When \( X \) is a zero mean random vector, that is, \( \overline{X}_i = 0 \) for every \( i = 1, 2, \ldots, k \),

\[
\lambda_{ij} = \overline{X}_i \overline{X}_j
\]

In that case,

\[
\Lambda_X = X^T X
\]

Note that

\[
X^T X = \begin{pmatrix}
X_1 \\
X_2 \\
\vdots \\
X_k
\end{pmatrix}
\begin{pmatrix}
X_1 & X_2 & \cdots & X_k
\end{pmatrix}
= \begin{pmatrix}
X_1X_1 & X_1X_2 & \cdots & X_1X_k \\
X_2X_1 & X_2X_2 & \cdots & X_2X_k \\
\vdots & \vdots & \ddots & \vdots \\
X_kX_1 & X_kX_2 & \cdots & X_kX_k
\end{pmatrix}
\]
Covariance Matrix of the Sum Vector

Let $X = (X_1, X_2, \ldots, X_k)$ be a zero-mean $k$-dimensional random vector.

Let $\Lambda_X$ denote the covariance matrix of $X$: $\Lambda_X = X^T X$.

Consider $N$ independent vectors $X_1, X_2, \ldots, X_N$ statistically identical to $X$.

Define the sum vector as

$$ Z = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} X_i. $$

Then

$$ \Lambda_Z = \Lambda_X \quad (g3) $$

**Proof**

Since $Z$ is a zero mean random vector,

$$ \Lambda_Z = Z^T Z $$

$$ = \frac{1}{N} \sum_{i=1}^{N} X_i^T \sum_{i=1}^{N} X_i $$

which is $(X_1^T + X_2^T + \cdots + X_N^T)(X_1 + X_2 + \cdots + X_N)$

$$ = \frac{1}{N} \left( \sum_{i=1}^{N} X_i^T X_i + \sum_{i=1}^{N} \sum_{j=1}^{N} X_i^T X_j \right) $$

noting $X_i^T X_j = X_i^T \overline{X_j} = 0$ for $i \neq j$

$$ \Lambda_Z = \frac{1}{N} \sum_{i=1}^{N} X_i^T X_i $$

$$ = \frac{1}{N} \sum_{i=1}^{N} \Lambda_X $$

$$ = \Lambda_X $$
Joint Characteristic Function of the Sum Vector

Let $Z = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} X_i$ and assume $\{X_i\}$ are iid to $X$.

Then

$$\ln \Phi_Z(\omega) = N \ln \Phi_X \left( \frac{\omega}{\sqrt{N}} \right)$$

**(g4)**

**Proof**

$$\Phi_Z(\omega) = e^{i\omega Z^T}$$

$$= e^{i\omega \frac{1}{\sqrt{N}} \sum_{i=1}^{N} X_i^T}$$

$$= \prod_{i=1}^{N} e^{i\omega \frac{1}{\sqrt{N}} X_i^T}$$

noting $\{X_i\}$ are independent

$$= \prod_{i=1}^{N} \Phi_{X_i} \left( \frac{\omega}{\sqrt{N}} \right)$$

noting $\{X_i\}$ are identical to $X$

$$= \left[ \Phi_X \left( \frac{\omega}{\sqrt{N}} \right) \right]^N$$
Joint CF of a zero-mean random vector

For a $k$-dim random vector $X$, its joint characteristic function is

$$\Phi_X(\omega) = e^{j\omega^TX}$$

where $\omega = (\omega_1, \omega_2, \cdots, \omega_k)$ and $X = (X_1, X_2, \cdots, X_k)$.

$$\omega X^T = \omega_1 X_1 + \omega_2 X_2 + \cdots + \omega_k X_k$$

Define the random variable $W$ as

$$W = j\omega X^T = j(\omega_1 X_1 + \omega_2 X_2 + \cdots + \omega_k X_k)$$

Then $\Phi_X(\omega) = e^{jW}$

$$= 1 + W + \frac{W^2}{2!} + \frac{W^3}{3!} + \cdots \quad (m1)$$

Now assume $X$ is a zero-mean random vector.

The 2nd term of eq.m1 is

$$\overline{W} = j(\omega_1 \overline{X_1} + \omega_2 \overline{X_2} + \cdots + \omega_k \overline{X_k})$$

For a zero-mean vector $X$,

$$\overline{X_j} = 0$$

and thus $\overline{W} = 0$.

3rd term of eq.m1:

$$\overline{W^2} = j^2 (\omega_1 \overline{X_1} + \omega_2 \overline{X_2} + \cdots + \omega_k \overline{X_k})^2$$

$$= - (\omega_1 \overline{X_1} + \omega_2 \overline{X_2} + \cdots + \omega_k \overline{X_k})(\omega_1 \overline{X_1} + \omega_2 \overline{X_2} + \cdots + \omega_k \overline{X_k})$$

$$= -\sum_{i=1}^{k} \sum_{j=1}^{k} \omega_i \overline{X_i X_j} \omega_j$$

recalling the covariance $\lambda_{ij} = \overline{X_i X_j}$ when $\overline{X_i} = \overline{X_j} = 0$

$$= -\sum_{i=1}^{k} \sum_{j=1}^{k} \omega_i \lambda_{ij} \omega_j$$

$$= -\omega^T A_X \omega \quad (m2)$$
Multi-dimensional Central Limit Theorem

Let $Z = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} X_i$ and $\{X_i\}$ be iid to $X$, where $\bar{X} = \theta$. Then

$$
\lim_{N \to \infty} \Phi_Z(\omega) = e^{-\frac{1}{2} \omega A_Z \omega^T} \quad (m3)
$$

where

$$
A_Z = A_X.
$$

$Z$ is referred to as a zero-mean Gaussian random vector when its joint characteristic function is the form shown in eq.m3.

Proof

From eq.m1 and m2,

$$
\Phi_X \left( \frac{\omega}{\sqrt{N}} \right) = 1 + \frac{\bar{W}}{\sqrt{N}} + \frac{\bar{W}^2}{2!N} + \frac{\bar{W}^3}{3!N^\frac{3}{2}} + \cdots \quad (m4)
$$

$$
= 1 - \frac{1}{2N} \omega A_X \omega^T + \frac{1}{N^\frac{3}{2}} f_3
$$

$$
\ln \Phi_X \left( \frac{\omega}{\sqrt{N}} \right) = \ln \left( 1 - \frac{1}{2N} \omega A_X \omega^T + \frac{1}{N^\frac{3}{2}} f_3 \right)
$$

Recalling $\ln(1 + u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \cdots; |u| < 1$

$$
\ln \Phi_X \left( \frac{\omega}{\sqrt{N}} \right) = -\frac{1}{2N} \omega A_X \omega^T + \frac{1}{N^\frac{3}{2}} f_3 + \text{other terms}
$$

From eq.m4,

$$
\ln \Phi_Z(\omega) = N \ln \Phi_X \left( \frac{\omega}{\sqrt{N}} \right) = -\frac{1}{2} \omega A_X \omega^T + \frac{1}{N^\frac{1}{2}} f_3 + \text{other terms}
$$

Finally

$$
\lim_{N \to \infty} \ln \Phi_Z(\omega) = -\frac{1}{2} \omega A_X \omega^T
$$

and from eq.m3, $A_Z = A_X$. 

Joint Char Function of non-zero mean Gaussian

Let $X$ be a Gaussian random vector with mean $m_X$ and covariance matrix $A_X$. Then its joint CF is

$$
\Phi_X(\omega) = e^{-\frac{1}{2} \omega A_X \omega^T + j \omega m_X^T}
$$

A Gaussian random vector is completely defined by the mean and its covariance matrix.

Proof.

Define $Y = X - m_X$.

Then $Y$ is a zero-mean Gaussian random vector, and it is easy to see $A_Y = A_X$.

From eq.m3,

$$
\Phi_Y(\omega) = \exp \left( -\frac{1}{2} \omega A_Y \omega^T \right).
$$

Thus

$$
\Phi_X(\omega) = \exp \left( j \omega X^T \right)
= \exp \left( j \omega (Y + m_X)^T \right)
= \exp \left( j \omega Y^T \right) \exp \left( j \omega m_X^T \right)
= \Phi_Y(\omega) \exp \left( j \omega m_X^T \right)
= \exp \left( -\frac{1}{2} \omega A_Y \omega^T + j \omega m_X^T \right)
\text{noting } A_X = A_Y
= \exp \left( -\frac{1}{2} \omega A_X \omega^T + j \omega m_X^T \right)
Formal Definition of Gaussian Random Vector

$X$ is a Gaussian random vector (or the component random variables are jointly Gaussian) if and only if its joint characteristic function is

$$
\Phi_X(\omega) = \exp \left( -\frac{1}{2} \omega \Lambda_X \omega^T + j \omega m_X^T \right)
$$

where $m_X$ is the mean vector and $\Lambda_X$ is the covariance matrix.

The joint pdf $f_X(x)$ can be found by the inverse Fourier transform:

$$
f_X(x) = \frac{1}{(2\pi)^{\frac{n}{2}} |\Lambda_X|^{\frac{1}{2}}} \exp \left( -\frac{1}{2} (x - m_X) \Lambda_X^{-1} (x - m_X)^T \right)
$$

Example

For $k = 1$, 
$X = X$
$m_X = [\mu]$
$\Lambda_X = [\sigma^2]$

$$
\Phi_X(\omega) = \exp \left( -\frac{1}{2} \omega \sigma^2 \omega + j \omega \mu \right)
$$
For $k = 2$,

$X = (X_1, X_2)$ jointly Gaussian with correlation coefficient $\rho$

$m_X = [\mu_1, \mu_2]$

$A_X = \begin{bmatrix}
    \lambda_{11} = \sigma_1^2 & \lambda_{12} = \text{cov}(X_1, X_2) \\
    \lambda_{21} = \text{cov}(X_2, X_1) & \lambda_{22} = \sigma_2^2
\end{bmatrix}$

recalling $\rho = \frac{\text{cov}(X_1, X_2)}{\sigma_1 \sigma_2}$

$= \begin{bmatrix}
    \sigma_1^2 & \rho \sigma_1 \sigma_2 \\
    \rho \sigma_1 \sigma_2 & \sigma_2^2
\end{bmatrix}$

$\Phi_X(\omega) = \exp \left( -\frac{1}{2} \omega A_X \omega^T + j \omega m_X^T \right)$

$\omega = [\omega_1, \omega_2]$

$\omega A_X \omega^T = [\omega_1, \omega_2] \begin{bmatrix}
    \sigma_1^2 & \rho \sigma_1 \sigma_2 \\
    \rho \sigma_1 \sigma_2 & \sigma_2^2
\end{bmatrix} \begin{bmatrix}
    \omega_1 \\
    \omega_2
\end{bmatrix}$

$= \omega_1^2 \sigma_1^2 + 2 \omega_1 \omega_2 \rho \sigma_1 \sigma_2 + \omega_2^2 \sigma_2^2$

$\omega m_X^T = [\omega_1, \omega_2] \begin{bmatrix}
    \mu_1 \\
    \mu_2
\end{bmatrix}$

$= \omega_1 \mu_1 + \omega_2 \mu_2$

$\Phi_X(\omega) = \exp \left( -\frac{\omega_1^2 \sigma_1^2 + 2 \omega_1 \omega_2 \rho \sigma_1 \sigma_2 + \omega_2^2 \sigma_2^2}{2} + j (\omega_1 \mu_1 + \omega_2 \mu_2) \right)$
The joint pdf of a Gaussian random vector is

\[ f_X(x) = \frac{1}{(2\pi)^{\frac{k}{2}} |A_X|^\frac{1}{2}} \exp\left(-\frac{1}{2} (x - m_X)^T A_X^{-1} (x - m_X)^T \right) \]

For \( k = 2 \),

\[ X = (X_1, X_2). \]

\[ x = (x_1, x_2). \]

\[ m_X = (\mu_1, \mu_2). \]

\[ x - m_X = (x_1 - \mu_1, x_2 - \mu_2). \]

\[ A_X = \begin{bmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{bmatrix}. \]

\[ |A_X| = \sigma_1^2 \sigma_2^2 - \rho^2 \sigma_1^2 \sigma_2^2 = \sigma_1^2 \sigma_2^2 (1 - \rho^2). \]

\[ A_X^{-1} = \frac{1}{|A_X|} \begin{bmatrix} \sigma_2^2 & -\rho \sigma_1 \sigma_2 \\ -\rho \sigma_1 \sigma_2 & \sigma_1^2 \end{bmatrix}. \]

\[ (x - m_X)^T A_X^{-1} (x - m_X) = \frac{\sigma_2^2 (x_1 - \mu_1)^2 - 2 \rho \sigma_1 \sigma_2 (x_1 - \mu_1)(x_2 - \mu_2) + \sigma_1^2 (x_2 - \mu_2)^2}{\sigma_1^2 \sigma_2^2 (1 - \rho^2)} \]

\[ = \frac{\left( \frac{x_1 - \mu_1}{\sigma_1} \right)^2 - 2 \rho \frac{x_1 - \mu_1}{\sigma_1} \frac{x_2 - \mu_2}{\sigma_2} + \left( \frac{x_2 - \mu_2}{\sigma_2} \right)^2}{(1 - \rho^2)} \]

Finally we have, for \( k = 2 \),

\[ f_{X_1, X_2}(x_1, x_2) = \frac{1}{2\pi \sigma_1 \sigma_2 (1 - \rho^2)^{\frac{1}{2}}} e^{-\frac{\left( \frac{x_1 - \mu_1}{\sigma_1} \right)^2 - 2 \rho \frac{x_1 - \mu_1}{\sigma_1} \frac{x_2 - \mu_2}{\sigma_2} + \left( \frac{x_2 - \mu_2}{\sigma_2} \right)^2}{2(1 - \rho^2)}} \]
Weighted Sum of Gaussian Random Variables

Let \( X \) be a Gaussian random vector and define \( Y \) as a linear transformation of \( X \)
\[
Y^T = AX^T + b^T
\]
where \( \text{dim } X = k \), \( A \) is a \( k \times k \) matrix, and \( b \) is a \( k \)-dimensional constant vector.
Then \( Y \) is also a Gaussian random vector with
\[
m_Y^T = Am_X^T + b^T \quad \text{and} \quad A_Y = AA_X A^T \quad (w1)
\]

Note. A sum of Gaussian random variables is Gaussian.
The component Gaussian random variables \( \{X_i\} \) don't have to be independent for the sum to be Gaussian.

Homework. Weighted Sum of Gaussian Random Variables

Prove that a transformation of a Gaussian random vector is a Gaussian random vector.

Hint.
Show \( \Phi_Y(\omega) = e^{-\frac{1}{2} \omega (AA_X A^T) \omega^T + i\omega (m_X A^T + b)^T} \) to prove \( Y \) is Gaussian with
\[
m_Y = m_X A^T + b \quad \text{and} \quad A_Y = AA_X A^T
\]

Note.
\( A \) can be a \( h \times k \) matrix with \( h < k \).
Eq.\( w1 \) still holds true.
Example:

Linear Combination of Gaussian Random Variables:

Assume \( X_1 \sim N(\mu_1, \sigma_1^2) \) and \( X_2 \sim N(\mu_2, \sigma_2^2) \)

Suppose \( X_1 \) and \( X_2 \) are jointly Gaussian with correlation coefficient \( \rho \).

Let

\[
Y_1 = a_1 X_1 + a_2 X_2 \\
Y_2 = X_1
\]

In this example,

\[
A = \begin{bmatrix}
  a_1 & a_2 \\
  1 & 0
\end{bmatrix}
\]

\( b = 0 \)

\( Y_1 \) and \( Y_2 \) are jointly Gaussian with

\[
m_Y^T = Am_X^T = \begin{bmatrix}
  a_1 & a_2 \\
  1 & 0
\end{bmatrix}
\begin{bmatrix}
  \mu_1 \\
  \mu_2
\end{bmatrix} = \begin{bmatrix}
  a_1 \mu_1 + a_2 \mu_2 \\
  \mu_1
\end{bmatrix}
\]

and

\[
A_Y = A A_X A^T = \begin{bmatrix}
  a_1 & a_2 \\
  1 & 0
\end{bmatrix}
\begin{bmatrix}
  \sigma_1^2 & \rho \sigma_1 \sigma_2 \\
  \rho \sigma_1 \sigma_2 & \sigma_2^2
\end{bmatrix}
\begin{bmatrix}
  a_1 & 1 \\
  a_2 & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
  a_1^2 \sigma_1^2 + 2a_1a_2 \rho \sigma_1 \sigma_2 + a_2^2 \sigma_2^2 & a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2 \\
  a_1 \sigma_1^2 + \rho a_2 \sigma_1 \sigma_2 & \sigma_1^2 + \rho \sigma_2 \sigma_1 \sigma_2
\end{bmatrix}
\]

Covariance matrix \( A_Y \) shows \( \text{VAR}(Y_1) = a_1^2 \sigma_1^2 + 2 \rho a_1 a_2 \sigma_1 \sigma_2 + a_2^2 \sigma_2^2 \)

Alternate method of finding \( \text{VAR}(Y_1) \):

\[
\text{VAR}(Y_1) = \text{VAR}(a_1 X_1 + a_2 X_2)
\]

\[
= \text{VAR}(a_1 X_1) + \text{VAR}(a_2 X_2) + 2 \text{COV}(a_1 X_1, a_2 X_2)
\]

\[
= a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2 + 2 \left( a_1 X_1 a_2 X_2 - a_1 \bar{X}_1 \cdot a_2 \bar{X}_2 \right)
\]

\[
= a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2 + 2 a_1 a_2 \left( \bar{X}_1 \bar{X}_2 - \bar{X}_1 \cdot \bar{X}_2 \right)
\]

\[
= a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2 + 2 a_1 a_2 \text{COV}(X_1, X_2)
\]

\[
= a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2 + 2 a_1 a_2 \rho \sigma_1 \sigma_2
\]
Suppose we are interested in $Y_1$ only.

Let

$$Y_1 = a_1 X_1 + a_2 X_2$$

In this example,

$$A = \begin{bmatrix} a_1 & a_2 \end{bmatrix}$$

The dimension of $Y$ can be smaller than that of $X$.

$Y_1$ is a Gaussian random variables with

$$m_Y^T = Am_X^T = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = a_1 \mu_1 + a_2 \mu_2$$

and

$$A_Y = A A_X A^T = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$= a_1^2 \sigma_1^2 + 2 \rho a_1 a_2 \sigma_1 \sigma_2 + a_2^2 \sigma_2^2$$

Covariance matrix $A_Y$ is $1 \times 1$, and shows $VAR(Y_1) = a_1^2 \sigma_1^2 + 2 a_1 a_2 \rho \sigma_1 \sigma_2 + a_2^2 \sigma_2^2$.
Multi-dimensional Central Limit Thm - Example k=3

\[ Z(t) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} X_i(t); \{X_i(t)\} \text{ are independent random telegraph signals} \]

\[ \{Z(t)\} \text{ are jointly Gaussian.} \]

As \( N \to \infty \), \( Z(t) \) becomes a Gaussian random process.

\[ \{Z(t_1),Z(t_2),Z(t_3)\} \text{ are jointly Gaussian.} \]
Covariance Matrix of the Random Telegraph Signal Samples

\( X(t) \) is a random telegraph signal with transition rate \( \alpha \) [transitions/second]

We have shown that \( X(t) \) is a WSS random process with

- mean \( m_X(t) = 0 \);
- variance \( \sigma_X^2(t) = X(t)^2 = 1 \);
- auto-correlation, \( R_X(\tau) = e^{-\alpha|\tau|} \)

In this example, we will take \( k = 3 \) time samples.

Sampling time instants are \( (t_1, t_2, t_3) = (1, 2, 3) \) [seconds].

\( X = (X_1, X_2, X_3) = (X(1), X(2), X(3)) \) is a 3-dimensional random vector.

\( \bar{X}_i = 0 \) for \( i = 1, 2, 3 \) or in vector notation, the mean vector \( m_X = 0 \).

\[ \lambda_{ij} = \text{cov}(X_i, X_j). \]

Since \( \bar{X}_i = \bar{X}_j = 0 \),

\[ \lambda_{ij} = \bar{X}_i \bar{X}_j = R_X(\bar{t}_j - \bar{t}_i) = e^{-\alpha|\bar{t}_j - \bar{t}_i|} \]

Let \( \Lambda_X \) be the covariance matrix of the random vector \( X \). Then

\[
\Lambda_X = \begin{pmatrix}
1 & e^{-2\alpha} & e^{-4\alpha} \\
e^{-2\alpha} & 1 & e^{-2\alpha} \\
e^{-4\alpha} & e^{-2\alpha} & 1
\end{pmatrix}
\]

Covariance Matrix of the Sum Vector

Define \( Z = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} X_i \).

Then \( \Lambda_Z = \Lambda_X \).
**Joint Characteristic Function**

\( X = (X_1, X_2, X_3) \) a 3-dimensional random vector

\[
\Phi_X (\omega_1, \omega_2, \omega_3) \triangleq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\omega_1 x_1} e^{i\omega_2 x_2} e^{i\omega_3 x_3} f_X (x_1, x_2, x_3) \, dx_1 \, dx_2 \, dx_3
\]

where \( f_X (x_1, x_2, x_3) \) is the joint pdf of \( X \).

Using expectation notation,

\[
\Phi_X (\omega_1, \omega_2, \omega_3) = e^{i(\omega_1 X_1 + \omega_2 X_2 + \omega_3 X_3)} \tag{el}
\]

Using matrix notation,

Let \( \omega = (\omega_1, \omega_2, \omega_3) \) and \( X = (X_1, X_2, X_3) \). Then

\[
\omega X^T = \omega_1 X_1 + \omega_2 X_2 + \omega_3 X_3 \quad \text{and eq.} \tag{el} \]

\[ \Phi_X (\omega) = e^{i\omega X^T} \]

For the sum vector,

\( Z = (Z_1, Z_2, Z_3) \) a 3-dimensional random vector

\[
\Phi_Z (\omega_1, \omega_2, \omega_3) \triangleq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\omega_1 z_1} e^{i\omega_2 z_2} e^{i\omega_3 z_3} f_Z (z_1, z_2, z_3) \, dz_1 \, dz_2 \, dz_3
\]

where \( f_Z (z_1, z_2, z_3) \) is the joint pdf of \( Z \).

Using matrix notation,

\[
\Phi_Z (\omega) = e^{i\omega Z^T}
\]

We do not know \( \Phi_X (\omega) \) yet.

However, as \( N \to \infty \), we can find \( \Phi_Z (\omega) \) without knowledge of \( \Phi_X (\omega) \).
Multi-dimensional Central Limit Theorem

As \( N \to \infty \),

\[
\Phi_Z(\omega) = \exp\left(-\frac{1}{2} \omega \mathbf{A}_Z \omega^T \right)
\]  \hspace{1cm} (e2)

where \( \mathbf{A}_Z = \mathbf{A}_X = \begin{pmatrix}
1 & e^{-2\alpha} & e^{-4\alpha} \\
e^{-2\alpha} & 1 & e^{-2\alpha} \\
e^{-4\alpha} & e^{-2\alpha} & 1
\end{pmatrix} \)

Eq.e2 is the joint characteristic function of a zero-mean Gaussian random vector.

Joint pdf of the Gaussian Random Vector

The joint pdf \( f_Z(z) \) can be found by the inverse Fourier transform from \( \Phi_Z(\omega) \):

\[
f_Z(z) = \frac{1}{(2\pi)^{\frac{3}{2}} |\mathbf{A}_Z|^2} \exp\left(-\frac{1}{2} z \mathbf{A}_Z^{-1} z^T \right) \quad \text{with} \ k = 3.
\]